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AUTHOR Carman, Robert A.
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ABSTRACT

This document discusses the value of linear programing in finding minimum and maximum solutions to problems of resource allocation. Three models using this technique are given for the areas of educational finance, school district personnel compensation, and instructional program evaluation. (RA)

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LINEAR PROGRAMMING

WITH

APPLICATIONS TO EDUCATIONAL PLANNING

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by

Robert A. Carman

SANTA BARBARA CITY COLLEGE

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LINEAR PROGRAMMING WITH APPLICATIONS TO EDUCATIONAL PLANNING

I. Introduction

In the Aeneid, Virgil describes what must be one of the earliest optimization problems recorded. Dido, a queen of Phoenicia, was forced to flee her homeland after her brother Pygmalion, King of Tyre, had murdered her husband for his wealth. Shipwrecked off the coast of Africa, near Tunis, she bargained with a local chieftan for as much land as could be surrounded by the hide of a bull. The hide was cut into long thin strips which were joined and laid in a semi-circle with the diameter on the sea coast. In this way, according to the legend, she obtained the maximum possible amount of land and founded the city of Carthage.

Dido's problem, the maximization of the area under a curve of given length, is one of several classic problems of mathematics that led to the calculus of variations. In the seventeenth century Isaac Newton attempted to determine the shape of the body that would experience minimum air resistance in flight. Later Daniel Bernoulli sought the shape of the path connecting two given points such that an object moving under gravity along this path would make the trip in minimum time. These problems have in common the optimization of a quantity associated with a family of curves. Dido examined a family of closed curves of given length and for this isoperimetric problem the associated number was the area bounded by the curve. For Newton the family of curves was a set of projectile shapes, while the associated number was the air resistance.

Bernoulli examined a family of curves joining the two given points and the associated number was the time required for an object to fall along this curve. The general problem of optimization is to select that choice, from a large number of interrelated choices, which maximizes or minimizes the value of some associated quantity.

An important class of optimization problem is that which requires the determination of the maximum or minimum value of a linear function defined over a polyhedral convex set of points. This situation is often met in problems of allocating limited resources to meet clearly defined goals. Mathematical programming is the name given to a set of techniques developed to solve this type of problem. Linear programming is the simplest, most widely used of these techniques. Other mathematical programming techniques include non-linear, integer, heuristic and dynamic programming. (1, 6, 9)

The general linear programming problem may be stated as follows.

Given a set of variables $X_1, X_2 \ldots X_n$ that characterize or describe a system under study, we may represent a goal or objective of the system as a linear function of these variables

$$P(X) = C_1 X_1 + C_2 X_2 + \ldots + C_n X_n \qquad (1)$$

P (X) is called the profit function or objective function and is a criterion of effectiveness of the system with respect to the objective under consideration.

The system must operate under certain limitations or constraints which may be expressed as a set of m linear inequalities in the variables

$X_1, X_2 \ldots X_n$.

$$
\begin{aligned}
a_{11} X_1 + a_{12} X_2 + \cdots + a_{1n} X_n &\leq b_1 \\
a_{21} X_1 + a_{22} X_2 + \cdots + a_{2n} X_n &\leq b_2 \\
&\vdots \\
a_{m1} X_1 + a_{m2} X_2 + \cdots + a_{mn} X_n &\leq b_n
\end{aligned} \qquad (2)
$$

In general, the variables $X_1, X_2 \cdots X_n$ are non-negative.

$$
\begin{aligned}
X_1 &\geq 0 \\
X_2 &\geq 0 \\
&\vdots \\
X_n &\geq 0
\end{aligned} \qquad (3)
$$

The constraint equation set Eq. 2 and non-negativity conditions Eq. 3 determine a convex set of points, the <u>feasible</u> set.

The problem is to maximize the objective function Eq. 1 subject to Eqns. 2 and 3. All points in the feasible set obviously satisfy the conditions imposed by Eqns. 2 and 3, and it may be shown that the optimum point desired is always located at one of the vertices of the feasible region.

The problem may be written in matrix notation.

If $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ an m x n matrix of coefficients

$X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$, an n x 1 column vector, $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$, an m x 1 column vector, and

$C = (c_1, c_2, \ldots c_n)$, a 1 x n row vector

Then the problem is to

maximize CX

subject to $AX \leq B$

and $X \geq 0$

Alternatively, it may be stated

minimize YB

subject to $YA \geq C$

and $Y \geq 0$

If the components of A, B, and C are the same, the second of these two problems is called the Dual of the first. Every linear programming problem, whether of the maximum or minimum type, has a dual as above. From the second problem above we see that

$y_i = \frac{\partial c_i}{\partial a_i}$ and y_i is the relative increase in cost incurred if the requirement for i increases. The dual of a problem has important economic meaning. (8)

We may solve simple linear programming problems visually by examining convex sets in the plane. However, for problems involving more than two variables it is unreasonable to depend on visualization of convex sets for the solution. Problems of more than two variables may be solved using a procedure, known as the Simplex Algorithm, that does not depend on graphical techniques. (7, 12)

These fundamental ideas may be made clearer by use of the following example.

A certain manufacturing plant employs 36 assemblers and 18 packers and produces two items, Dohickeys and Thigs. A Dohickey requires one unit

of assembly time and one unit of packing time while a Thig requires two units of assembly time and only one-half unit of packing time. Each Dohickey sold returns a profit of $1.50 while each Thig earns $1.00. The market is such that we can sell all of these items that are produced. What are the rates of daily production of Dohickeys and Thigs that would maximize profits?

Let D = daily production rate of Dohickeys

T = daily production rate of Thigs

Daily Profit = $1.5\,D + 1.0\,T$

If each employee works 8 hours per day,

$D + 2\,T \leq 36 \times 8$ or $D + 2\,T \leq 288$

$D + \frac{1}{2}\,T \leq 18 \times 8$ or $D + \frac{1}{2}\,T \leq 144$

and the problem may be restated as:

maximize	$P = 1.5\,D + 1.0\,T$	The objective function
subject to	$D + 2\,T \leq 288$	The constraint set
	$D + \frac{1}{2}\,T \leq 144$	
and	$D \geq 0$	
	$T \geq 0$	The non-negativity conditions

The feasible set is shown graphically in Fig. 1.

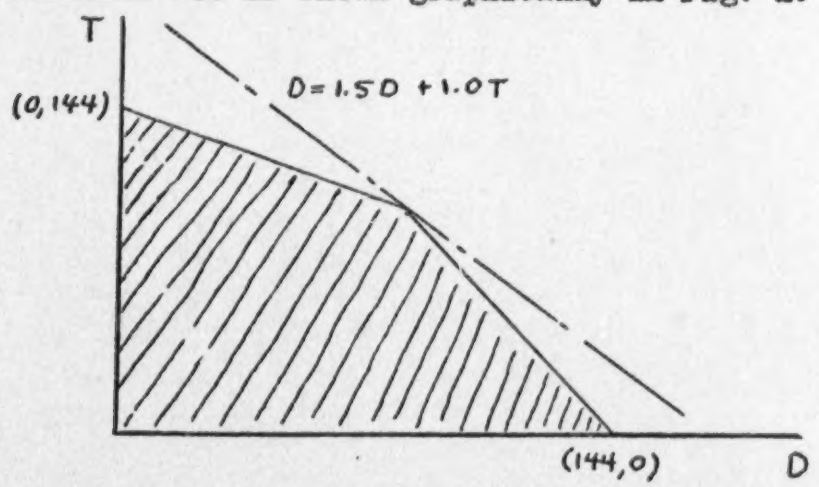


Fig. 1.

The optimum solution is at the vertex (96, 96) and the maximum profit is

$$P_{max} = 1.5 \times 96 + 1.0 \times 96 = \$240.$$

While linear programming is a powerful tool useful in a wide variety of optimization problems, the assumptions and limitations of the procedure must be made explicit.

1. The objective function must be definite and quantitative. The analyst must be able to express the situation in mathematical terms and this usually involves a need to make underlying assumptions explicit and to formulate the problem in terms of certain key variables. The selection or creation of these variables and process of fitting them into a comprehensive descriptive system is known as mathematical modelling. (11)

2. A systematic way must be found to describe quantitatively the limitations or constraints under which the system operates. This assumes that the variables or activities specified in 1. are interrelated and that they can, in fact, be identified and quantified. In any complex situation this may be difficult to do, and the analyst may instead work with those aspects of the situation that are readily quantifiable. There is a danger here that more subtle and perhaps equally important factors may be ignored.

3. All variables, both in the objective function and constraint equations, must appear in linear combination. This restriction is not so limiting as it may seem, since a very large number of the models used in economics and planning methodology are roughly linear. The cost,

in whatever units, of producing n items is often linearly dependent on n.

Linear programming models reflect complex situations with many linearly interacting variables, all competing for fixed resources, and with some linear criterion of effectiveness whose maximization or minimumization is required. The technique is particularly valuable for the solution of resource allocation problems in educational planning. Sections II, III and IV of this paper will explore specific educational applications.

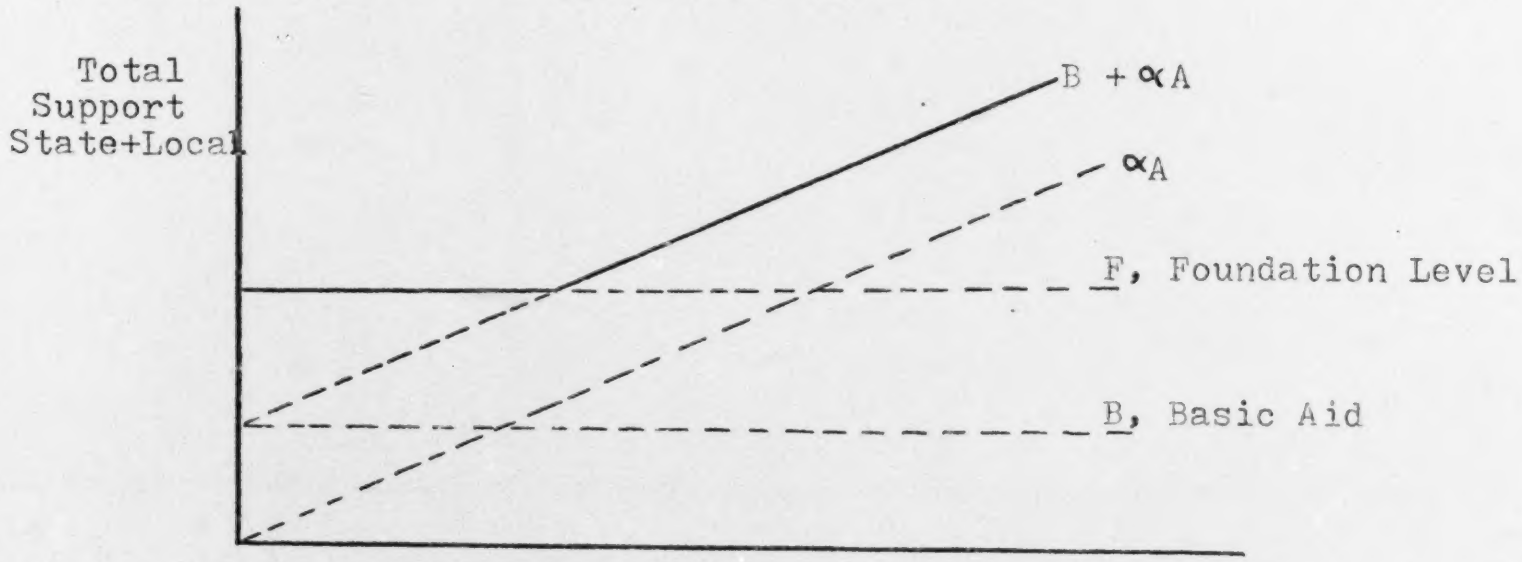
II. School Finance Programs (2, 5)

A basic problem in school finance is the development of schemes by which state resources can be allocated to support educational programs in local school districts. The administration and derivation of these schemes is carried out at state level by means of simplistic formulas that mathematically define the relationships between state and local funding. In order to overcome inadequacies in the basic scheme, supplementary programs are added until the school finance scheme takes on a patchwork appearance. Any advantages that might have been gained through use of the original simple scheme are quickly outweighed by the awkwardness, confusion and ineffectiveness of such a multi-simplistic formula.

From a systems viewpoint, all resources available to support state and local educational expenditures, all local activities which consume these resources, and all pertinent political and economic considerations should be considered simultaneously. With our present knowledge of operations research techniques and availability of computers, simplicity

should not be the over-riding factor in resource allocation systems design. School finance must be viewed as a complex total system of resource allocation in a constrained environment. One attempt in this direction, a linear programming model based on the fiscal structure of a foundation-type state support program, is described here.

The relationships between state and local funding in foundation-type state support programs is illustrated in Fig. 2.



Y_i = total state aid per ADA to district i

A_i = assessed valuation per ADA in district i

F = Foundation level of the program, a minimum guaranteed level of expenditure

$\propto$ = state-mandated uniform tax rate which all districts must levy in order to qualify for the program.

B = uniform basic state aid per ADA for the program

If $B + \propto A_i \geq F$ total state aid per ADA to district i

$$= Y_i = B \tag{4}$$

If $B + \propto A_i < F$ total state aid per ADA to district i

$$= Y_i = F - \propto A_i \tag{5}$$

Note that all districts in the state program, regardless of local assessed valuation per ADA, receive the uniform state basic aid B. Furthermore, all districts are required to maintain a minimum level of effort to support educational expenditures determined by a state-mandated tax rate $\propto$. The district must tax at this minimum rate in order to qualify for state funding. Because districts differ in assessed valuation the sum of basic aid and local funds $B + \propto A$ does not always reach the foundation level, a minimum guaranteed level of expenditure per student. Districts with low assessed valuation per ADA receive equalization aid $F - (B + \propto A)$ designed to correct this imbalance.

As a result of this foundation-type program, there exists a considerable spread in total education support between districts. As a result the per pupil expenditure varies widely between the highest and lowest local ability districts. This spread may be expected to result in discernible differences in educational opportunity throughout the state. The more affluent districts are better able to compete for teacher talent, provide higher teacher salaries, provide support facilities and specialist personnel, etc.

A linear programming model of a state support system may be constructed as follows (2):

Maximize F, the foundation level,

$$F = Y_i - E_i + \propto A_i \tag{6}$$

E_i is the expenditure per ADA in excess of the foundation level

Total local funds used = L

$$L = \sum \alpha A_i \; N_i \tag{7}$$

Where N_i is the ADA in district i

Total state funds used = S = $\sum Y_i \; N_i$ (8)

Total overall costs of the state support program = T

$$= S + L \tag{9}$$

$$= \sum \alpha A_i N_i + \sum Y_i N_i \tag{10}$$

$$= \sum N_i (\alpha A_i + Y_i) \tag{11}$$

Relationships between total local and state shares of the total costs may be established

$L \leq a_1 \; T$ $\quad a_1$ = maximum local share of total cost (12)

$L \geq a_2 \; T$ $\quad a_2$ = maximum local share of total cost (13)

$S \leq b_1 \; T$ $\quad b_1$ = maximum state share of total cost (14)

$S \geq b_2 \; T$ $\quad b_2$ = maximum state share of total cost (15)

The spread in total district expenditure per ADA, between the lowest local ability and highest local ability district in the system, is a very important design criterion for a state support program. This spread may be controlled in the model by specifying that

$$E_1 \leq c \; F \tag{16}$$

Where C = maximum percentage of the foundation level at optimum that the "lowest local ability" district can differ from the final total district expenditure in the "highest local ability" district in the system.

If C = 0 all districts in the system will have the same final total district expenditure per ADA

If C equals some constant, a fixed percentage spread in final total district expenditure will be permitted.

In this model we will assume that C = 0 and

$$E_i = 0 \tag{17}$$

If student tuition may be charged or other funds are available, a new resource must be added to the system.

Tuition funds or other funds raised in district i = $\sum Z \; N_i$

Where Z = tuition cost or extra funding per student

In this model we shall assume that Z = 0 (18)

and that the total state funds available S is a constant.

Applying the model to the California Junior College State Support Program, constraint parameters may be obtained from Table I.

TABLE I

Variable or Parameter	Current Value in the System	Values Allowed in the Model	
		Upper Bound	Lower Bound
F, Foundation level ($ per ADA)	600	800	500
α, Tax rate ($ per $10,000 assessed valuation per ADA)	25	50	5
Y_i, Basic or minimum aid to each district ($ per ADA)	125	800	0
S, Total state costs (millions)	36.917	36.917	36.917
T, Total overall costs (millions)	101.143	None	None
C, Spread in final total district expenditure per ADA ($ per ADA)	119	0	0
Z, Tuition or extra funding ($ per ADA)	0	0	0

The linear programming problem may be stated:

Maximize F

subject to

$$A_i \alpha + Y_i - F - E_i = 0$$

$$E_i = 0$$

$$\sum N_i \; Y_i = S$$

$$\sum N_i \alpha A_1 = L$$

$$S + L = T$$

$$S \leq 0.500\ T$$

$$S \geq 0.365\ T$$

$$L \leq 0.635\ T$$

$$L \geq 0.500\ T$$

$$F \geq 500$$

$$F \leq 800$$

$$\alpha \geq 5$$

$$\alpha \leq 50$$

$$Y_i \geq 0$$

$$Y_1 \leq 800$$

$$Z = 0$$

$$W = 36.917$$

and

$$Y_i \geq 0$$

$$E_i \geq 0$$

$$S \geq 0$$

$$L \geq 0$$

$$T \geq 0$$

Table II lists the values of certain variables and parameters both at optimum and at their former value.

TABLE II

Variable or Parameter	Value at Optimum	Former Value
F, Foundation level ($ per ADA)	615	600
C, Spread in district expenditure per ADA ($ per ADA)	0	119
α, Tax rate ($ per $10,000 assessed valuation per ADA)	25	25
S, Total state funds used (millions)	36.917	36.917
L, Local funds used (millions)	101.143	101.143
Minimum aid to each district ($ per ADA)	20	125

Without the use of additional state funds a maximum foundation level of $615 per ADA was maintained. State funds obviously had to be transferred from the highest to the lowest local ability districts. A total of 37 of the 42 districts received more state aid under the model while only five (all from the ten highest local ability districts) received less state aid.

One severe limitation of the model in its present form is its apparent neglect of the actual financial needs of a district. This limitation is shared by simplistic models. If an equation of the form

$$N_i \quad Y_i \geq D_i \tag{19}$$

Where D_i = financial needs of the district broken down into costs for pupil transportation, teacher salaries, compensatory education programs, etc.

Y_i = amount of state funds per ADA allocated to district i

N_i = average daily attendance in district i

This would guarantee each district in the system the minimum amount of state funds required to operate certain aspects of its educational program, the remainder of these funds being distributed in accordance with the objective function used.

III. Compensation of School District Personnel (4, 5)

The most widely used method of compensation for school district personnel is the fixed step salary schedule. Fixed step increments in salary are associated with years in the school system. Shifts in salary classification are made on the basis of amount of formal training as indicated by advanced degrees, college units or inservice training completed. Typically the fixed step salary schedule considers only these two factors in the determination of compensation for teaching personnel.

There is general agreement that this simplistic approach is inadequate and fails to consider many factors of importance.

1. Internal Consistency. Each employee of the school district should receive a fair and adequate salary in relation to every other employee. In the fixed step schedule maximum and minimum salaries, intermediate step size, and formal training increases are usually arbitrarily determined.

2. Relative Difficulty of Assignment. Many educators feel that salaries should in some way reflect the relative difficulty of the instructional assignment or environment.

3. District Resources. The fixed step salary schedule does not consider the extent of available school district resources in determining salaries. Available resources must be used to determine salaries in a logical and consistent way.

4. Salary Hierarchies. It is a common practice in many districts for the salaries of school administrators to be linked with the teacher's salary schedule. Administrator salaries may be expressed as a ratio to some aspect of the teachers' schedule, or in terms of an absolute salary differential. These practices tend to maintain the gap between these groups. Clearly, overlaps should be permitted.

5. Personnel Availability. In order for the school district to compete economically for teachers and administrators possessing special skills, it should be possible to include salary differentials into the salary scheme used.

6. Multiple Factors in Evaluation. An effective salary scheme should consider many factors for evaluation, including awards or recognition for good teaching, administrative responsibilities, inservice training, teacher characteristics, etc.

7. School District Priorities. Any salary scale should reflect school district policies and priorities if district resources are to be expended in a manner that advances the district toward its goals. The district should be able to easily modify its salary scheme to place the greatest weight on whatever aspect of its operation seems to best advance it toward pre-set goals. For example, it may wish to maximize starting salaries so as to attract the most competent new teachers, to

ensure program continuity by maximizing salary highs, or seek change in disadvantaged areas by maximizing the salaries of teachers assigned there.

The following salary scheme utilizes a linear programming approach in an attempt to satisfy all of these factors in a single dynamic model. (4)

Ten variables are included in the model:

X_1 = Type of teaching environment. This variable may take on three values:

3 = Difficult

2 = Medium

1 = Easy

X_2 = Priority of subject matter area. This variable may take on three values:

3 = High Priority

2 = Medium Priority

1 = Low Priority

X_3 = Supervisory responsibility of the employee. This variable may take on seven values:

7 = Single district wide (superintendent)

6 = District wide

5 = Single school wide (principle)

4 = School wide

3 = Department wide (Department head)

2 = Class wide (teacher)

1 = Student (teachers aide)

X_4 = Highest academic degree attained. This variable may take on five values:

5 = Doctorate

4 = MA, MS

3 = M.Ed

2 = BA

1 = AA

X_5 = Total work experience. This variable may take on 7 values:

7 = 12 + years

6 = 10 to 12 years

5 = 8 to 10 years

4 = 6 to 8 years

3 = 4 to 6 years

2 = 2 to 4 years

1 = 0 to 2 years

X_6 = Earned special distinction for teaching. This variable may take on only two values:

2 = With distinction

1 = Without distinction

X_7 = Number of college units or semester hours completed beyond the highest degree attained. This variable may take on 8 values:

8 = 36 + units

7 = 31 to 35 units

6 = 26 to 30 units

5 = 21 to 25 units

4 = 16 to 20 units

3 = 11 to 15 units

2 = 6 to 10 units

1 = 0 to 5 units

X_8 = Number of inservice hours of credit. This variable may take on 7 values:

7 = 30 + hours

6 = 26 to 30 hours

5 = 21 to 25 hours

4 = 16 to 20 hours

3 = 11 to 15 hours

2 = 6 to 10 hours

1 = 0 to 5 hours

X_8 = 0 for personnel not directly concerned with the instructional process.

X_9 = Relative additional workload type. This variable may take on 5 values.

5 = District wide

4 = School wide

3 = Department wide

2 = Class wide

1 = Student

X_{10} = Special bonus for personnel at the highest salary for their respective function. This variable may take on five values:

5 = Superintendents

4 = Administrators

3 = Department Heads

2 = Teachers

1 = Teacher Aides

Five job classifications may be considered:

j = 1 Represents Superintendent

j = 2 Represents Administrator (principals, vice principals, etc.)

j = 3 Represents Department Heads

j = 4 Represents Teachers

j = 5 Represents Teacher aides and other teacher assistance personnel such as lab assistants and readers.

It is possible to describe each job classification by means of two equations; one represents the theoretically most highly qualified person for the job classification, and hence the highest salary, and the other represents the theoretically lowest qualifications, hence the lowest salary. These equations are:

$$a_1 X_1 + a_2 X_2 + \cdots + a_{10} X_{10} = H_j \qquad (20)$$

$$b_1 X_1 + b_2 X_2 + \cdots + b_9 X_9 = L_j \qquad (21)$$

a_i = The highest rated characteristics associated with variable X_i appropriate to job classification j.

b_i = The lowest rated characteristics associated with variable X_i appropriate to job classification j. Note that variable X_{10} is not included in the determination of lowest salary.

H_j = The theoretically highest salary to be paid in job classification j.

L_j = The theoretically lowest salary to be paid in job classification j.

Salary differences may be represented by the equation

$$H_j - H_{j+1} \leq D \tag{22}$$

Where H_j = The theoretically highest salary in job classification j.

H_{j+1} = The theoretically highest salary in the next lower job classification j + 1

D = The specified difference in salary.

Salary overlaps between various job classifications may be represented by

$$L_j \geq W_1 \; H_{j+1} \tag{23}$$

$$L_j \leq W_2 \; H_{j+1} \tag{24}$$

Where

W_1 is the desired minimum percentage overlap between H_{j+1}, the theoretically highest salary in job classification j + 1 and L_j, the theoretically lowest salary in job classification j.

W_2 is the desired maximum percentage overlap between job classification j and j + 1.

Budgetary constraints are incorporated into the model through the equation

$$\sum_{i,k} N_{ik} \; V_{ik} \; X_i \leq T \tag{25}$$

Where N_{ik} = The number of district employees having characteristic k on factor i.

V_{ik} = The relative value of characteristic k on factor i.

X_i = The factor i used in salary evaluation.

T = The total amount of school-district resources available for salaries.

The 33 equations (Eq. 20-25), plus upper and lower bounds on X_i, L_j and H_j, represent system constraints that may be specified by determining the values of a_i, b_i, H_j, L_j, D, W_1, W_2, N_{ik}, V_{ik}, and T.

The objective function represents a criterion of effectiveness selected by the school district in the light of its established goals and objectives. Several possibilities are obvious:

1. Maximize beginning teacher salaries I_4
2. Maximize highest teacher salaries H_4
3. Maximize salary weight for factor X_1, eg. pay higher salaries to teachers in difficult assignments such as inner city schools.

Using data from a typical school system, the linear programming problem may be stated as follows:

Maximize X_1

subject to $3X_1 + 3X_2 + 7X_3 + 5X_4 + 5X_5 + 2X_6 + 5X_7 + 5X_9 + 5X_{10} = H_1$

$$X_1 + X_2 + 6X_3 + 3X_4 + 2X_5 + X_6 + X_7 + 5X_9 = L_1$$

$$3X_1 + 3X_2 + 5X_3 + 5X_4 + 5X_5 + 2X_6 + 5X_7 + 4X_9 + 4X_{10} = H_2$$

$$X_1 + X_2 + 4X_3 + 3X_4 + X_5 + X_6 + X_7 + 4X_9 = L_2$$

$$3X_1 + 3X_2 + 3X_3 + 5X_4 + 7X_5 + 2X_6 + 8X_7 + 7X_8 + 3X_9 + 3X_{10} = H_3$$

$$X_1 + X_2 + 3X_3 + 2X_4 + X_5 + X_6 + X_7 + X_8 + 3X_9 = L_3$$

$$3X_1 + 3X_2 + 2X_3 + 5X_4 + 7X_5 + 2X_6 + 8X_7 + 8X_8 + 2X_9 + 2X_{10} = H_4$$

$$X_1 + X_2 + 2X_3 + 2X_4 + X_5 + X_6 + X_7 + X_8 + 2X_9 = L_4$$

$$3X_1 + 3X_2 + X_3 + 2X_4 + 5X_5 + 2X_6 + 5X_7 + 6X_8 + X_9 + X_{10} = H_5$$

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8 + X_9 = L_5$$

$L_1 \geq 0.50\ H_1$ (spread of salaries within job classifications)

$L_1 \leq 0.80\ H_1$

$L_2 \geq 0.50\ H_2$

$L_2 \leq 0.75\ H_2$

$L_3 \geq 0.50\ H_3$

$L_3 \leq 0.75\ H_3$

$L_4 \geq 0.50\ H_4$

$L_4 \leq 0.75\ H_4$

$L_5 \geq 0.50\ H_4$

$L_5 \leq 0.75\ H_4$

$H_2 \geq 1.05\ L_1$ (salary overlap between job classifications)

$H_2 \leq 1.20\ L_1$

$H_3 \geq 1.05\ L_2$

$H_3 \leq 1.20\ L_2$

$H_4 \geq 1.10\ L_3$

$H_4 \leq 1.30\ L_3$

$H_5 \geq 1.10\ L_4$

$H_5 \leq 1.30\ L_4$

$H_1 - H_2 \geq 3500$ (absolute salary spreads)

$H_2 - H_3 \geq 3000$

$H_3 - H_4 \geq 2500$

$H_4 - H_5 \geq 2000$

$$2992 X_1 + 2873 X_2 + 2892 X_3 + 6126 X_4 + 5432 X_5 + 1422 X_6 + 5736 X_7 + 3950 X_8 + 2880 X_9 + 284 X_{10} = \$13,500,000$$ (total funds available for salaries)

$L_6 \geq 4500$ (limitation on lowest salary in the district)

$H_1 \leq 23{,}500$ (limitation on top salary in the district)

$X_1 \leq 1000$ (bounds on salary weights for each factor)

$X_1 \geq 100$

$X_2 \leq 1000$

$X_2 \geq 100$

$X_3 \leq 1500$

$X_3 \geq 100$

$X_4 \leq 1000$

$X_4 \geq 100$

$X_5 \leq 500$

$X_5 \geq 100$

$X_6 \leq 1500$

$X_6 \geq 100$

$X_7 \leq 500$

$X_7 \geq 100$

$X_8 \leq 500$

$X_8 \geq 100$

$X_9 \leq 2000$

$X_9 \geq 100$

$X_{10} \leq 1000$

$X_{10} \geq 100$

Solution of the problem gives the optimum salary weights X_i as shown in Table III.

TABLE III

Factor	Description	Value
X_1	Difficulty of Instructional Environment	$ 495.83
X_2	Subject Matter Priority	$ 100.00
X_3	Supervisory Responsibility	$ 1500.00
X_4	Highest Academic Degree	$ 100.00
X_5	Work Experience	$ 100.00
X_6	Special Awards or Distinction	$ 1365.63
X_7	Additional Academic Credits	$ 100.00
X_8	Additional Inservice Units	$ 100.00
X_9	Relative Additional Workload	$ 1138.54
X_{10}	Extra Bonus for Long Service	$ 161.45

Table IV indicates the highest and lowest salaries for the optimum solution obtained.

TABLE IV

	Job Description	Salary
j = 1a	Superintendent	$ 22,757 to $ 17,257
j = 2a	Administrators	$ 18,618 to $ 13,015
j = 3	Department Heads	$ 15,618 to $ 10,377
j = 4	Teachers	$ 12,918 to $ 7,738
j = 5	Teacher Aides	$ 9,118 to $ 5,000

Individual salaries S may be found from

$$S = \sum_{i,j} Y_{ij} \; X_i$$

Where Y_{ij} = the relative weight of characteristic j found on factor i possessed by the individual.

X_i = the optimal weight for factor i obtained from the linear programming problem solution.

For example, a teacher ($2X_3$) with a masters degree ($4X_4$) teaching in a difficult learning area ($3X_1$) with five years experience as a teacher ($3X_5$), with no special distinction ($1X_6$) or subject matter priority ($2X_2$), with 10 additional college units ($2X_7$) and 15 inservice units ($2X_8$) with normal additional workload ($2X_9$) and no bonus pay (X_{10}) would expect to earn the following salary:

$$3X_1 + 2X_2 + 2X_3 + 4X_4 + 3X_5 + 1X_6 + 2X_7 + 2X_8 + 2X_9 = I_4$$

Using the optimum salary weights given in Table III,

$$3\,(496) + 2\,(100) + 2\,(1500) + 4\,(100) + 3\,(100) + 1\,(1365) + 2\,(100) + 2\,(100) + 2\,(1139) = \$\,9431$$

In a similar manner, the salaries for each member in the school district may be determined.

This model may be profitably used as a framework for discussion in collective bargaining or other salary negotiations with organized teacher groups. In this way, demands specified by these groups may be examined to determine their internal logic and consistency. Discussion ideally would center on the determination of model parameters and the impact of parameter changes on the system could be made readily apparent.

IV. Net-Shift Analysis (3, 10)

The traditional method of evaluating instructional programs is to compare the final achievement score of some experimental group with the initial performance of the same group, with the performance of control groups, or with some standard target score. This type of evaluation is concerned only with changes in mean scores and ignores changes in the overall distribution of student performance scores. The statistical techniques employed give a static dichotomous assessment of student gains in achievement.

Education's current concern with accountability, evident in performance contracting ventures, new programs to meet special needs, and the systems approach to instruction, demands more dynamic evaluative tools. Net shift analysis is an application of linear programming that provides a continuous, dynamic, quantitative measure of the shift in distribution of achievement scores with respect to some goal distribution. The net shift approach provides a means of evaluating the effect of instruction on special sectars of the distribution such as lowest decile students or high achievers. It is especially useful as a means of determining the compensation of performance contractors. Under a compensation scheme based on net shift analysis schools set a target distribution and the goal of the contract is to produce an improved distribution of scores. The analysis provides a quantitative continuous measure of the degree to which the goal has been met.

It seems evident that if changes in the mean score of a group of students is the sole basis of a contract, the contractor will concentrate his efforts on those individuals who show the greatest potential gain in achievement. Therefore it is important in designing a contract to consider

the differential difficulty of shifting students from one region of the distribution to another. For example, it may be more difficult to shift fifth graders from the third to the fourth decile on standard reading tests than from the first to the third decile. If the contractor is aware of this his attention will focus on the easy transition where greatest achievement gain can be had. Obviously, school district and contractor utility differ.

To perform a net shift analysis, three distributions must be defined:

1. The Initial distribution I is the distribution of scores existing at the start of instruction in terms of standardized intervals. The distribution may be most conveniently written as a row vector. For example, we may describe the distribution in terms of the fraction above and below the norm $I = (.67, .33)$, in Quartiles $I = (.37, .32, .21, 10)$, or in deciles $I = (.04, .06, .10, .20, .20, .10, .10, .10, .05, .05)$.

2. The Achieved or Post-instructional distribution P is the distribution of scores found after the new instructional program has been in effect for a certain period of time. Continuing the example above, P may be

   $P = (.56, .44)$, $P = (.20, .25, .30, .25)$ in quartiles, or

   $P = (0, 0, 0, .10, .25, .30, .10, .10, .10, .05)$ in deciles.

3. The Goal distribution G defines the expectations or aspirations of the school district. This distribution could be based on city, state, or national goals or norms, on special instructional emphases, or on historical evidence.

In the example above, G might be

$$G = (.50, .50) \text{ or } G = (.25, .75)$$

$$G = (0, .25, .50, .25) \text{ in quartiles,}$$

$$\text{or } G = (0, 0, 0, 0, .10, .25, .20, .20, .15, .10).$$

A Utility or reward matrix U defines the school district utility associated with each of the possible transitions inherent in the score distributions. If the score distribution is specified in terms of n standardized intervals, that is, as an n-dimensional row vector, then U is an n x n matrix.

$$\text{If } I = (I_1, I_2, I_3 \cdots I_n)$$

$$\text{Then } U = \begin{pmatrix} U_{11} & U_{12} & U_{13} & \cdots & U_{1n} \\ U_{21} & U_{22} & U_{23} & \cdots & U_{2n} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ U_{n1} & U_{n2} & U_{n3} & \cdots & U_{nn} \end{pmatrix}$$

Where U_{ij} is the utility associated with a transition from score interval i to score interval j.

In general, the elements of U may be calculated as

$$U_{ij} = D_{ij} \; R_{ij} \tag{26}$$

Where D_{ij} are the elements in a perceived degrees of difficulty matrix D.

D_{ij} measures the difficulty of a transition from score interval i to score interval j.

R_{ij} is the perceived school district reward or utility associated with a transition from interval i to interval j.

These matrices may be obtained from historical evidence, teacher observations, administrator judgments, etc. Specifically, the D_{ij} may be taken as the reciprocals of the probability of the ij transition, based on actual records. For example, if, on the basis of past experience, we know the probability of transitions above and below the norm to be

	below norm	above norm
Below norm	.8	.2
Above norm	.3	.7

That is, in a given semester 80% of the students below the norm in reading stay below the norm, and so on.

Then

$$D = \begin{pmatrix} \frac{1}{.8} & \frac{1}{.2} \\ \frac{1}{.3} & \frac{1}{.7} \end{pmatrix} = \begin{pmatrix} 1.25 & 5.00 \\ 3.33 & 1.43 \end{pmatrix}$$

If $R = \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$

That is, the desired transition from below to above the norm is given utility 2, undesired transitions from above to below the norm are given utility -1, a penalty, and so on.

Then, by Eq. 26

$$U_{ij} = D_{ij} R_{ij}$$

so that

$$U = \begin{pmatrix} 0 & 10 \\ -3.33 & 1.43 \end{pmatrix}$$

Finally, two transition matrices T_1 and T_2 are defined.

T_2 is a matrix describing the transition from the initial distribution I to the post-instruction distribution P.

$$P = I \cdot T_2 \tag{27}$$

$$T_2 = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & & \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{pmatrix}$$

Where the X_{ij} are transition probabilities. (28)

Because T_2 is a transition probability matrix,

$$\begin{aligned} X_{11} + X_{12} + \cdots + X_{1n} &= 1 \\ X_{21} + X_{22} + \cdots + X_{2n} &= 1 \\ &\vdots \\ X_{n1} + X_{n2} + \cdots + X_{nn} &= 1 \end{aligned} \tag{29}$$

T_1 is a matrix describing the transition from the initial distribution I to the goal distribution G.

$$G = I \cdot T_1 \tag{30}$$

$$T_1 = \begin{pmatrix} X_{11}' & X_{12}' & \cdots & X_{1n}' \\ X_{21}' & X_{22}' & & X_{2n}' \\ \vdots & & & \\ X_{n1}' & X_{n2}' & \cdots & X_{nn}' \end{pmatrix} \tag{31}$$

Again, the X_{ij}' are transition probabilities, so that,

$$\begin{aligned} X_{11}' + X_{12}' + \cdots + X_{1n}' &= 1 \\ X_{21}' + X_{22}' + \cdots + X_{2n}' &= 1 \\ &\vdots \\ X_{n1}' + X_{n2}' + \cdots + X_{nn}' &= 1 \end{aligned} \tag{32}$$

The net school district utility associated with the transition $I \longrightarrow P$ is

$$Z = \sum X_{ij} \; U_{ij} \tag{33}$$

The net school district utility associated with the transition $I \longrightarrow G$ is

$$Z' = \sum X'_{ij} \; U_{ij} \tag{34}$$

Then, the ratio

$$A = \frac{Z}{Z'} \tag{35}$$

is a quantitative assessment of the degree to which the goal has been met, starting from the given initial distribution. This ratio may be used in determining appropriate compensation for a performance contractor producing distribution transition I-P with respect to some goal distribution G and given school district utility U.

It is important to emphasize that the X_{ij} and X'_{ij} are not the actual transitions, but are the hypothetical transitions needed to maximize net Utility Z and Z'. X_{ij} and X'_{ij} are obtained from the solution to the following linear programming problems:

I. Maximize $Z = X_{ij} \; U_{ij}$ (33)

subject to
$$\begin{aligned} X_{11} + X_{12} + \cdots + X_{1n} &= 1 \\ X_{21} + X_{22} + \cdots + X_{2n} &= 1 \\ &\vdots \\ X_{n1} + X_{n2} + \cdots + X_{nn} &= 1 \end{aligned} \tag{29}$$

and
$$I \cdot T_2 = P \tag{27}$$

$$\text{i.e.}\quad \begin{aligned} I_1 X_{11} + I_2 X_{21} + \cdots + I_n X_{n1} &= P_1 \\ I_1 X_{12} + I_2 X_{22} + \cdots + I_n X_{n2} &= P_2 \\ &\vdots \\ I_1 X_{1n} + I_2 X_{2n} + \cdots + I_n X_{nn} &= P_n \end{aligned}$$

and the non-negativity conditions

$$X_{ij} \geq 0$$

II. Maximize $$Z' = \sum X'_{ij}\ U_{ij} \qquad (34)$$

subject to $$\begin{aligned} X'_{11} + X_{12}' + \cdots + X'_{1n} &= 1 \\ X'_{21} + X'_{22} + \cdots + X'_{2n} &= 1 \\ &\vdots \\ X'_{n1} + X'_{n2} + \cdots + X'_{nn} &= 1 \end{aligned} \qquad (32)$$

and $$I \cdot T_1 = G \qquad (30)$$

$$\text{i.e.}\quad \begin{aligned} I_1 X_{11}' + I_2 X'_{21} + \cdots + I_n X'_{n1} &= G_1 \\ I_1 X'_{12} + I_2 X'_{22} + \cdots + I_n X'_{n2} &= G_2 \\ &\vdots \\ I_1 X_{1n}' + I_2 X'_{2n} + \cdots + I_n X'_{nn} &= G_n \end{aligned}$$

and the non-negativity conditions

$$X'_{ij} \geq 0$$

The following example (3) may serve to clarify the process:

Given $$I = (.663, .337)$$

$$P = (.575, .425)$$

$$G = (.500, .500)$$

If $$D = \begin{pmatrix} \frac{1}{.8} & \frac{1}{.2} \\ \frac{1}{.3} & \frac{1}{.7} \end{pmatrix} = \begin{pmatrix} 1.25 & 5.00 \\ 3.33 & 1.43 \end{pmatrix}$$

and

$$R = \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix}$$

Then

$$U = \begin{pmatrix} 0 & 10 \\ -3.33 & 1.43 \end{pmatrix}$$

$$X_{11}' \geq 0$$
$$X_{21}' \geq 0$$
$$X_{12}' \geq 0$$
$$X_{22}' \geq 0$$

The solution to this problem is

$$X_{11}' = 0.754$$
$$X_{12}' = 0.245$$
$$X_{21}' = 0$$
$$X_{22}' = 1.000$$

giving maximum $Z' = 387.85$

Then

$$A = \frac{Z}{Z'} = \frac{274.72}{387.85} = 0.708$$

We may interpret this value of A to mean that, in the light of school district Utility U, 70.8% of the goal has been met. Under a performance contract the contractor would receive $70.80 per pupil per $100 of goal value.

Obviously, other values of the utility matrix may be used, changing the coefficients in Equations 27, 30, 33, and 34 of the linear programming problem. Actual transitions of students are not measured, instead the transitions required to maximize net school district utility are calculated and used in the evaluation of the instructional results.

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